ELEN E3401: Electromagnetics

Spring 2025

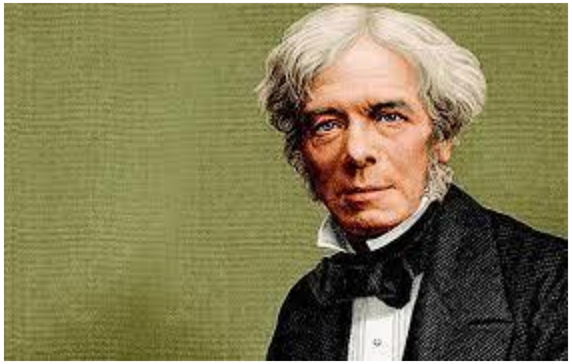
Prof. Keren Bergman

Lecture #17

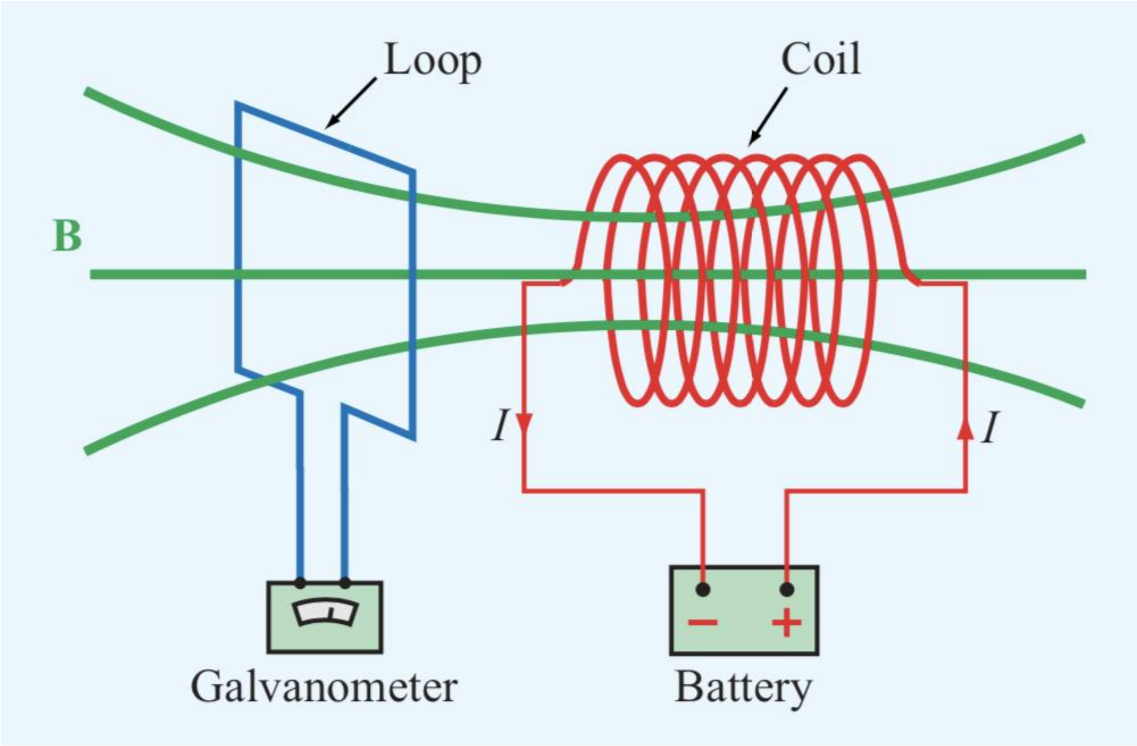
COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Faraday’s law

- Oersted established wire carrying current produces a magnetic field that exerts a magnetic force (compass needle)
- Faraday hypothesized that if current can produce a magnetic field, the magnetic field should produce a current in the wire
- Numerous experiments by Michael Faraday (UK) and Joseph Henry (NY) – 10 years



Faraday's law

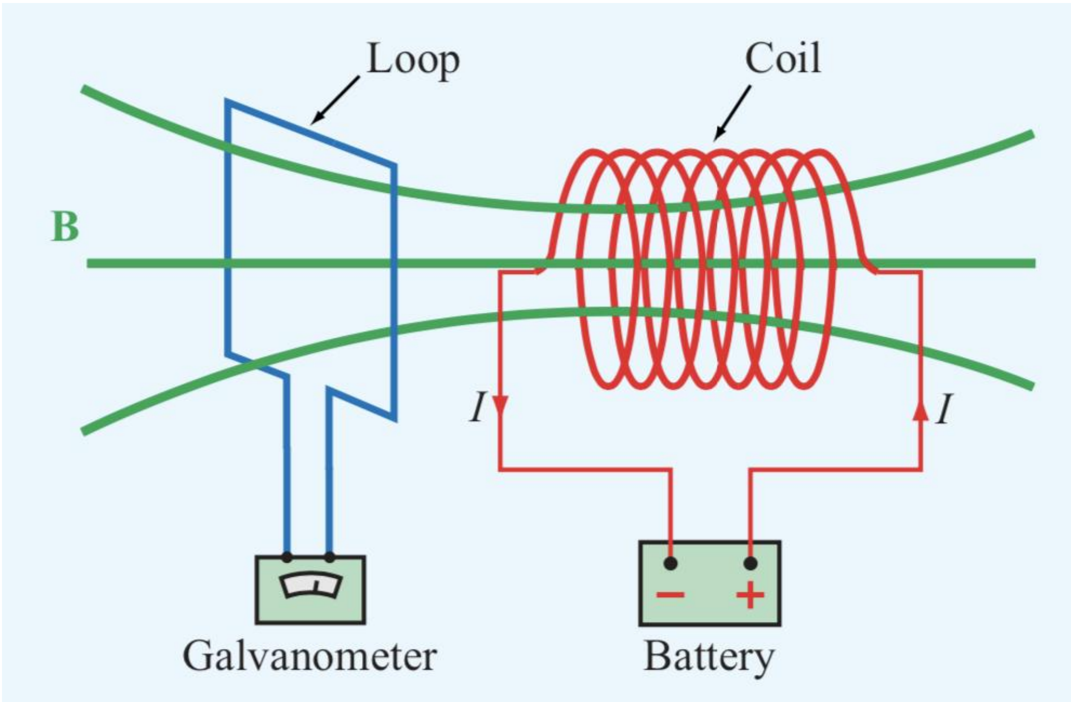


A conducting loop connected to a galvanometer (instrument to detect current flow) is placed next to a conducting coil connected to a battery

Current in the coil produces a magnetic field $\vec{B}$ → lines pass through the loop (magnetic flux)

$$Manetic\ Flux = \Phi = \int_S \vec{B} \cdot d\vec{s}$$

Faraday’s law



Manetic Flux

$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$

Initial measurements → no current

DC current → constant $\vec{B}$ → constant flux → no current

Battery connect/disconnect → interrupt current flow → magnetic flux changes → momentary deflection of galvanometer needle

<u>Changes in magnetic flux </u>induces a current

Electromotive Force (emf)

flow of current through the coil, voltage induced across the terminals

Electromotive force (emf) → V_{emf}

emf induced in a closed conducting loop of N turns:

$$V_{emf} = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} \quad [V]$$

Results were obtained by Faraday and Henry independently
but attributed to Faraday, with **Faraday's Law**

Three Types of EMF

$$V_{emf} = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} \quad [V]$$

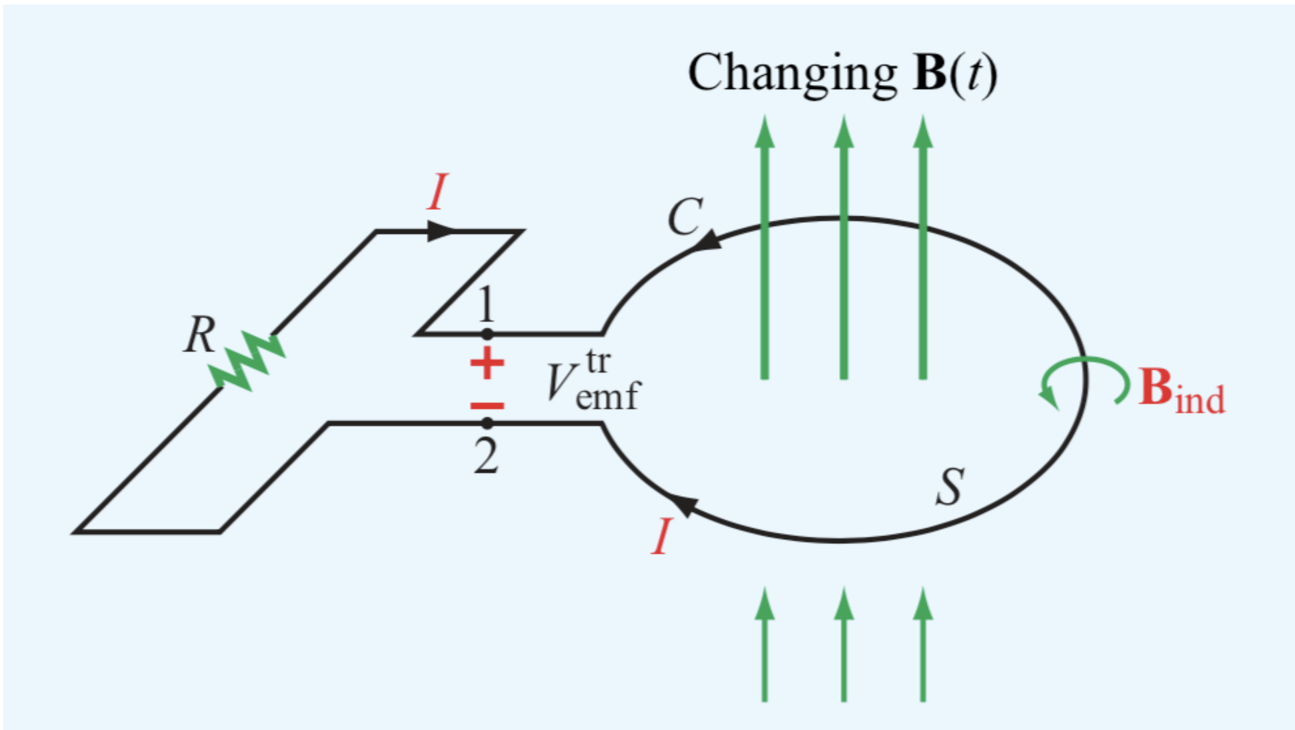
- Derivative is <u>total time derivative</u> on $\vec{B}$ *and* $d\vec{s}$
- emf can be generated in a closed conducting loop by any of the following 3 ways:

1. Time-varying magnetic field linking stationary loop → transformer emf: V_{emf}^{tr}
2. A moving loop with a time-varying surface area (relative to the normal component of $\vec{B}$) in a static field $\vec{B}$ → motional emf: V_{emf}^{m}
3. A moving loop in a time-varying field $\vec{B}$

The total emf is given by: $V_{emf} = V_{emf}^{m} + V_{emf}^{tr}$

Stationary Loop in a Time-Varying Magnetic Field: V_{emf}^{tr}

Consider a stationary loop (S is constant) in a time varying $\vec{B}$

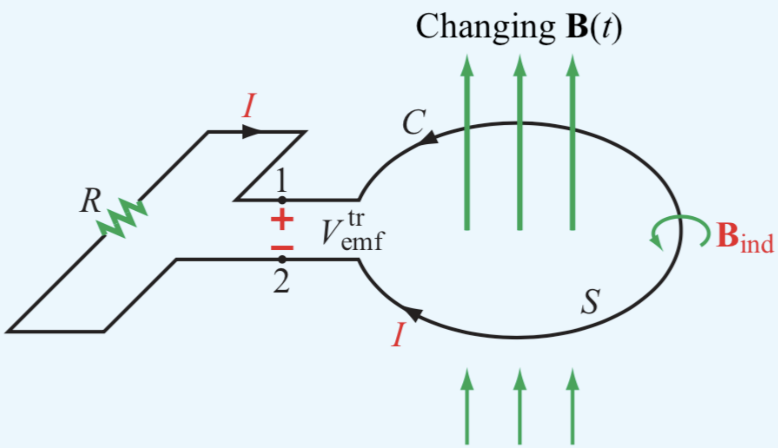


$$V_{emf}^{tr} = -N\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} = -N\int_S \frac{\partial}{\partial t}\vec{B}\cdot d\vec{s}$$

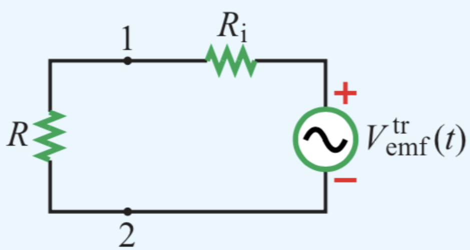
Since loop is <u>stationary</u>, *d/dt* only operates on $\vec{B}(t)$

Stationary Loop in a Time-Varying Magnetic Field

Consider a stationary loop (S is constant) in a time varying $\vec{B}$



(a) Loop in a changing **B** field



(b) Equivalent circuit

If loop has internal resistance R_i:

$$I = \frac{V_{emf}^{tr}}{R + R_i}$$

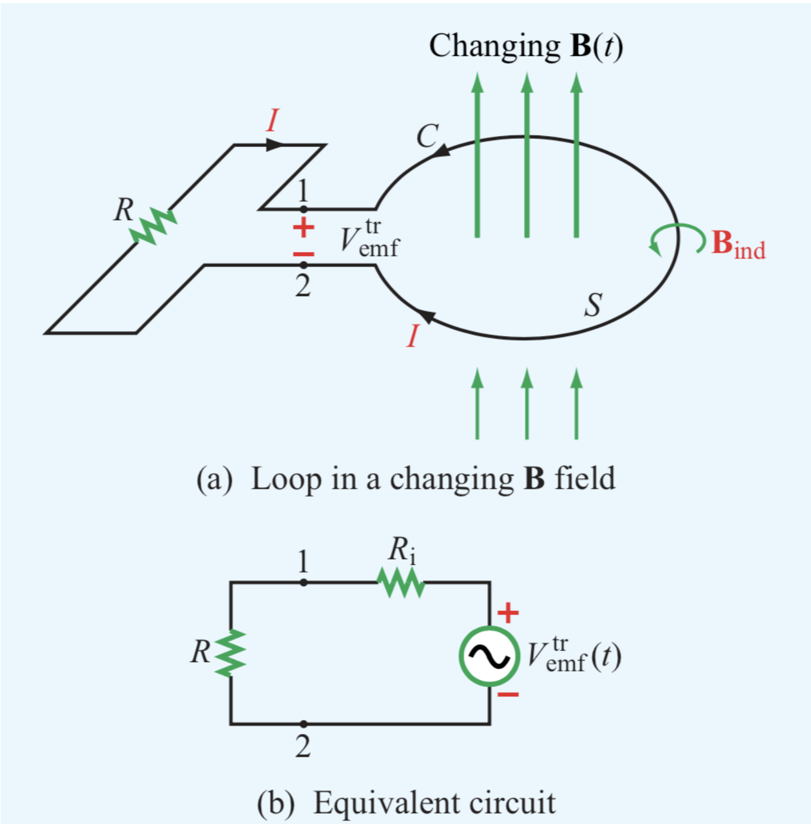
Polarity of V_{emf}^{tr} / direction of current is governed by **Lenz's law**:

I in the loop is always in a direction opposing the change of magnetic flux that produced I

I induces $\vec{B}_{ind}$ with corresponding Φ_{ind}

$\vec{B}_{ind}$ serves to oppose the *change* in $\vec{B}(t)$

Stationary Loop in a Time-Varying Magnetic Field



Treat the loop as a closed path with contour *C* (ignore small circuit opening):

$$V_{emf}^{tr} = \oint_C \vec{E} \cdot d\vec{l}$$

For loop N=1

$$V_{emf}^{tr} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

Closed loop integral, apply Stoke's theorem:

$$\oint_C \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{s}$$

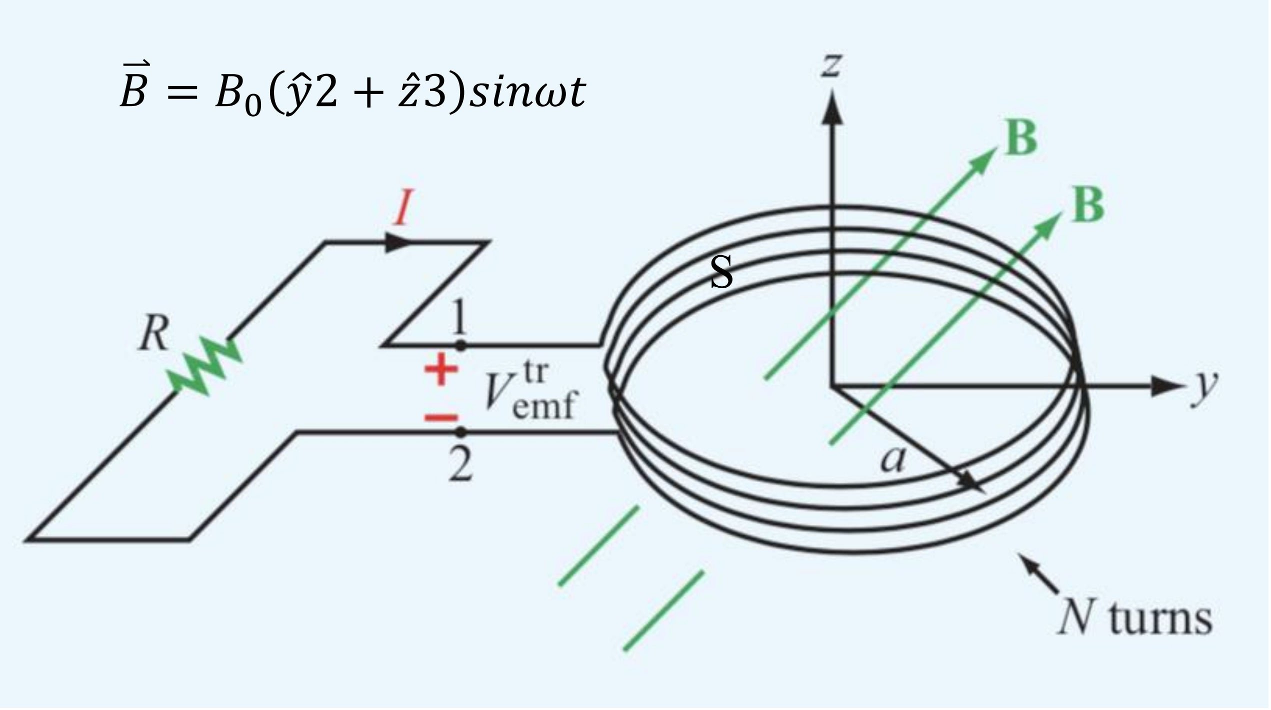
$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (Faraday's\ Law)}$$

Example: inductor in a changing magnetic field

Inductor formed by N turns of conducting wire radius = *a*

→ Inductor loop in *x*-*y* plane, centered at origin, connected to resistor *R*

→ Magnetic field $\vec{B} = B_0(\hat{y}2 + \hat{z}3)sin\omega t$

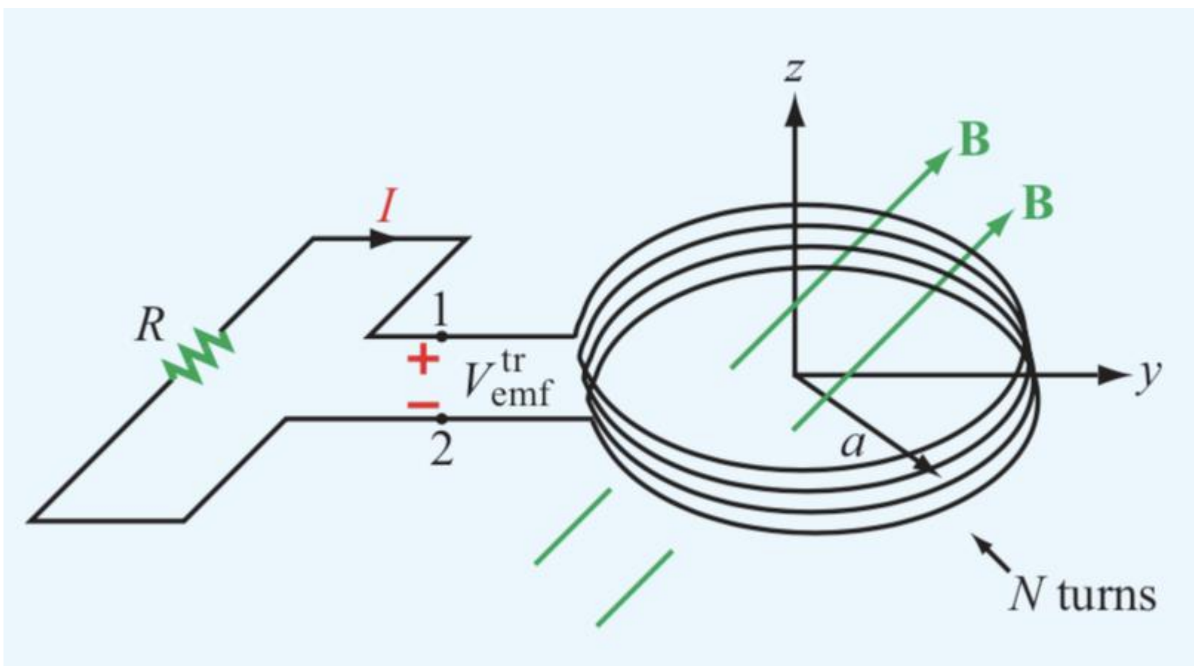


Example: inductor in a changing magnetic field

a) Obtain magnetic flux linking a single turn

Magnetic flux: $\Phi = \int_S \vec{B} \cdot d\vec{s} = \int_S [B_0(\hat{y}2 + \hat{z}3) sin\omega t] \cdot \hat{z}ds$

$$\Phi = 3\pi a^2 B_0 sin\omega t \; [Wb]$$



Example: inductor in a changing magnetic field

b) Obtain the transformer emf (V_{emf}^{tr}): N = 10, $B_0 = 0.2T, a = 10$cm, $\omega = 10^3 rad/s$

$$V_{emf}^{tr} = -N\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \qquad \text{Transformer emf}$$

$$V_{emf}^{tr} = -N\frac{d\Phi}{dt} = -\frac{d}{dt}3\pi N a^2 B_0 sin\omega t = -3\pi N\omega a^2 B_0 cos\omega t$$

For N = 10, $B_0 = 0.2T, a = 0.1$m, $\omega = 10^3 rad/s$

$$V_{emf}^{tr} = -188.5cos10^3 t \quad [\text{V}]$$

Example: inductor in a changing magnetic field

c) Obtain the polarity of V_{emf}^{tr} at t = 0

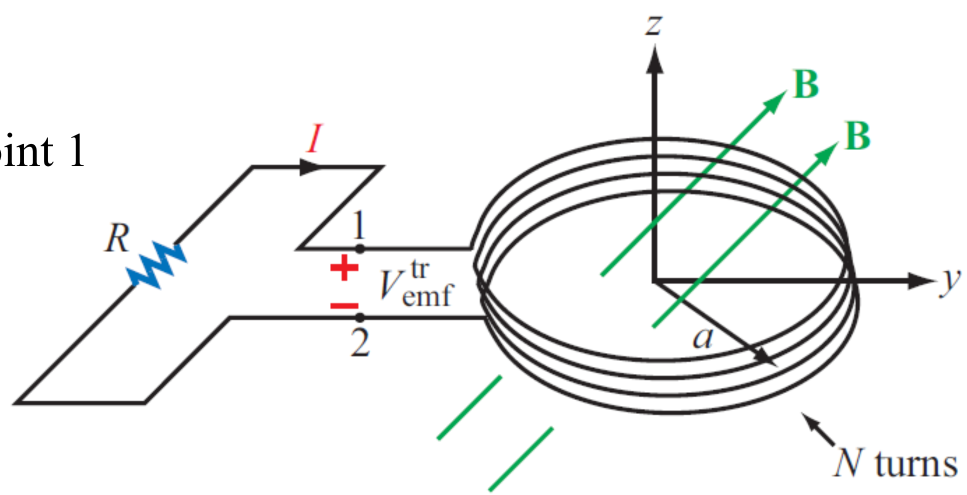
Magnetic flux ($\Phi = 3\pi a^2 B_0 sin\omega t$) is increasing and $\vec{B}$ in + $\hat{z}$

At $t = 0$, $\frac{d\Phi}{dt} > 0$ and $V_{emf}^{tr} = -188.5V$

Since flux is increasing, current flowing in loop to satisfy Lenz's law – is in a direction opposing the change of magnetic flux that produced I

Point 2 at higher potential than point 1

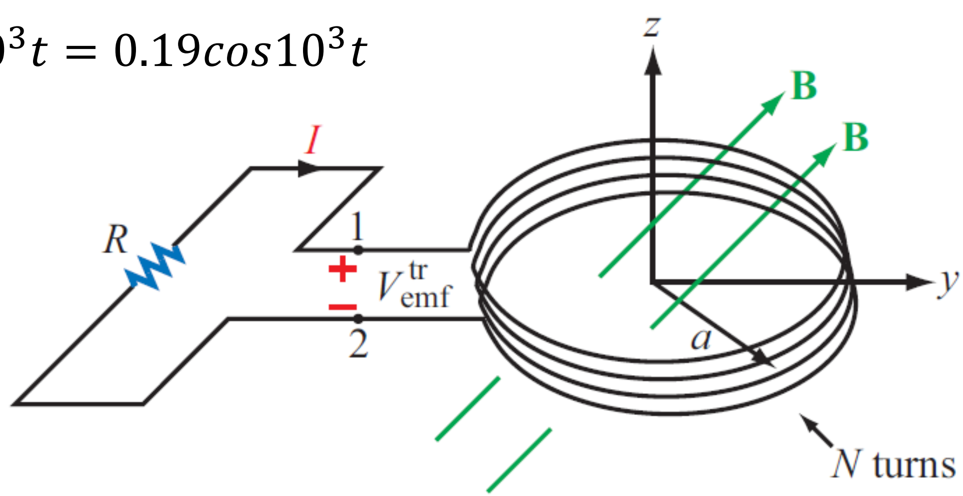
$$V_{emf}^{tr} = V_1 - V_2 = -188.5V$$



Example: inductor in a changing magnetic field

d) Obtain induced current for R = 1kΩ (inductor wire resistance << R)

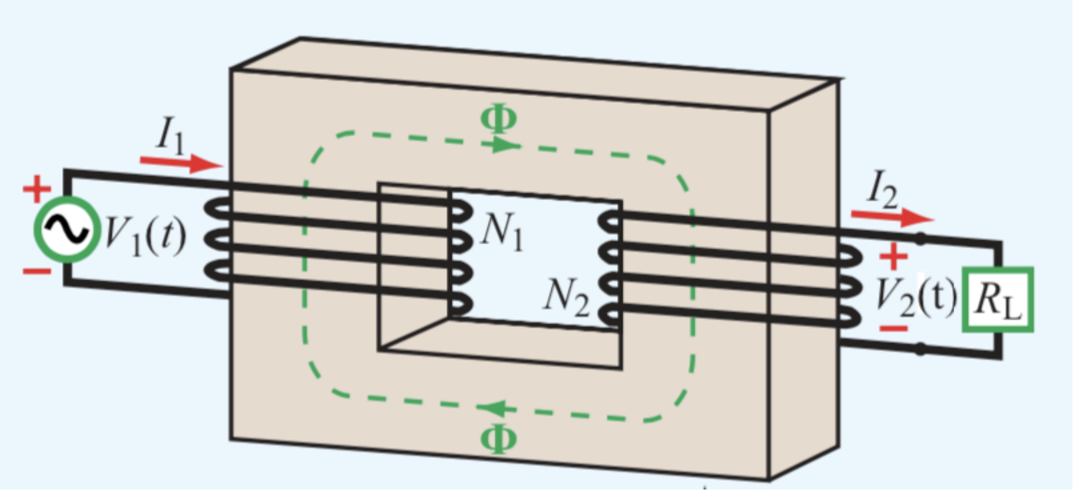
$$I = \frac{V_2 - V_1}{R} = \frac{188.5}{10^3} cos10^3 t = 0.19 cos10^3 t$$



Extra: what is V_{emf}^{tr} if $\vec{B} = \hat{y} B_0 cos\omega t$?

$V_{emf}^{tr} = 0$ → since $\vec{B}$ is perpendicular to loop surface normal, $d\vec{s}$

Ideal Transformer



Primary coil has N_1 turns, connected to $V_1(t)$

Second coil has N_2 turns, connected to load R_L

Ideal transformer → $\mu = \infty$

Primary side of transformer:

V_1 generates current I_1 in primary coil, establishing flux Φ in magnetic core.

Flux Φ and voltage V_1 related by **<u>Faraday's Law:</u>** $V_1 = -N_1 \dfrac{d\Phi}{dt}$

For secondary side: $V_2 = -N_2 \dfrac{d\Phi}{dt}$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

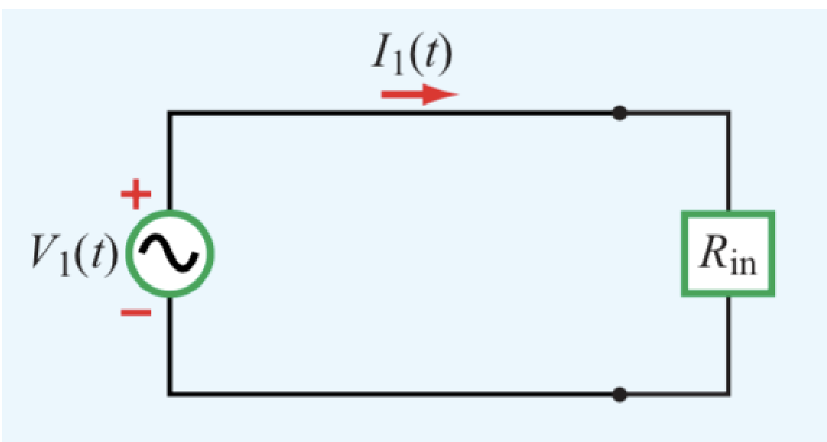
Ideal Transformer

Ideal, lossless transformer: all instantaneous power delivered to load by primary source coil → no power lost in core

$P_1 = P_2 \qquad P_1 = I_1V_1 \qquad P_2 = I_2V_2 \qquad \frac{I_1}{I_2} = \frac{N_2}{N_1}$ (Opposite of voltage ratio)

$\frac{N_1}{N_2} = 0.1$ → V_2 would be 10X V_1 but I_2 is $\frac{I_1}{10}$

$V_2 = I_2R_L$ → equivalent circuit of transformer to input:



$$R_{in} = \frac{V_1}{I_1} = \frac{V_2}{I_2}\left(\frac{N_1}{N_2}\right)^2 = R_L\left(\frac{N_1}{N_2}\right)^2$$

If load is Z_L and V_1= sinusoidal
Phase domain equivalent:

$$Z_{in} = Z_L\left(\frac{N_1}{N_2}\right)^2$$